Fuzzy Decomposition on the Affine Kac-Moody Algebras $F_4^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ & $G_2^{(1)}$

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Abstract—The theory of Kac- Moody algebras is one of the modern field of Mathematical research which has been developing rapidly in the past twenty years due to its interesting connections and applications to other fields of Mathematics and Mathematical Physics. On the other hand fuzzy theory has deep rooted applications in varies fields of science and technology. We make an attempt on studying the fuzziness on Kac-Moody algebras. In this paper, we define fuzzy sets on the Cartesian product of some of the affine type of Kac-Moody algebras $F_4^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ & $G_2^{(1)}$. Basic properties of fuzzy sets are studied; For specific values of α, α - level and strong α - level sets are computed. α -cut decomposition for there fuzzy sets, associated with the $F_4^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ & $G_2^{(1)}$ families of affine type of Kac-Moody algebras are computed.

Index Terms— Affine type, α -cut decomposition, α -level sets, convexity, fuzzy set, Kac-Moody algebra, root basis

1 INTRODUCTION

1.1 Basic definitions on Kac-Moody algebras

Definition 1.1. [3] An integer matrix $A=(a_{ij})_{i,j=1}^{n}$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

i. $a_{ii} = 2 \ \forall \ i = 1,2,...,n$

ii. $a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \ \ i, j = 1,2,...,n$

iii. $a_{ij} \le 0 \ , i \ne j \ \forall \ i, j = 1,2,...,n$

Let us denote the index set of A by N = {1,…,n}. A GCM A is said to decomposable if there exist two non-empty subsets $I, J \subset N$ such that $I \cup J = N$ and $a_{ij} = a_{ji} = 0 \ \forall \ i \in I$ and $j \in J$. If A is not decomposable, it is said to be indecomposable.

Definition 1.2. [2] A realization of a matrix $A=(a_{ij})_{i,j=1}^{n}$ is a triple (H, Π, Π^{v}) where l is the rank of A, H is a $2n-l$ dimensional complex vector space, $\Pi = \{\alpha_1, ..., \alpha_n\}$ and $\Pi^{v} = \{\alpha_1^{v}, ..., \alpha_n^{v}\}$ are linearly independent subsets of H^* and H respectively, satisfying $\alpha_j(\alpha_i^{v}) = a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are called simple roots. The root lattice generated by Π is $Q = \sum_{i=1}^{n} Z\alpha_i$.

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Definition 1.3. [2] The Kac-Moody algebra g(A) associated with a GCM $A=(a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i, f_i, $i = 1,2,...,n$ and H with the following defining relations :

$$\begin{aligned}
[h, h'] &= 0, \quad h, h' \in H \\
[e_i, f_j] &= \delta_{ij}\alpha_i^{v} \\
[h, e_j] &= \alpha_j(h) e_j \\
[h, f_j] &= -\alpha_j(h) f_j \ , \ i, j \in N \\
(ad\ e_i)^{1-a_{ij}} e_j &= 0 \\
(ad\ f_i)^{1-a_{ij}} f_j &= 0 \quad \forall \ i \ne j, \ i, j \in N
\end{aligned}$$

The Kac-Moody algebra g(A) has the root space decomposition $g(A) = \underset{\alpha \in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A) = \{x \in g(A) / [h, x] = \alpha(h)x, \ for \ all \ h \in H\}$. An element α, $\alpha \ne 0$ in Q is called a root if $g_\alpha \ne 0$. Let $Q_+ = \sum_{i=1}^{n} Z_+ \alpha_i$. Q has a partial ordering "$\le$" defined by $\alpha \le \beta$ if $\beta - \alpha \in Q_+$, where $\alpha, \beta \in Q$.

Definition 1.4. [2] Let $\Delta (= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Definition 1.5. [2] To every GCM A is associated a Dynkin diagram S(A) defined as follows: S(A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|, |a_{ji}|\}$ number of lines if $a_{ij} \, . \, a_{ji} \le 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $a_{ij} \, . \, a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|, |a_{ji}|)$ of integers.

Theorem 1.6. [2] Let A be a real n x n matrix satisfying (m1), (m2) and (m3).

(m1) A is indecomposable;

(m2) $a_{ij} \le 0, \quad i \ne j;$

(m3) $a_{ij} = 0$ implies $a_{ji} = 0$

Then one and only one of the following three possibilities hold for both A and ${}^tA$:

(i) det A $\ne 0$; there exists u > 0 such that A u > 0; Av ≥ 0 implies v > 0 or v $= 0$;

(ii) co rank A=1; there exists u > 0 such that Au $= 0$; Av ≥ 0 implies Av $= 0$;

(iii) there exists u > 0 such that Au < 0; Av ≥ 0, v ≥ 0 imply v $= 0$.

Then A is of finite, affine or indefinite type iff (i), (ii) or (iii) is satisfied.

Definition 1.7. [5] A Kac- Moody algebra g(A) is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.

We note that for the affine type of Kac-Moody algebra the rank of the GCM $A = n-1$. $i.e., l = n-1$.

For a detailed study on Kac Moody algebras one can refer to [2].

1.2 Basic definitions on fuzzy sets

Definition 1.8. [6] A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$ that can be finite, countable or over countable.

Definition 1.9. [6] If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ is defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / x \in X\}$. $\mu_{\tilde{A}}(x)$ is called the membership function or "grade of membership" of x in $\tilde{A}$ that maps X to the membership space M.

Definition 1.10. [6] The support of a fuzzy set $\tilde{A}$, $S(\tilde{A})$ is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

Definition 1.11. [6] The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-level set $A_\alpha = \{x \in X / \mu_{\tilde{A}}(x) \ge \alpha\}$; $A_\alpha{}' = \{x \in X / \mu_{\tilde{A}}(x) > \alpha\}$ is called "Strong α - level set" or "Strong α - cut".

Definition 1.12. [6] Let $\tilde{A}$ be a fuzzy set on X. Then the set $\{x \in X / \mu_{\tilde{A}}(x) = 1\}$ is called the core of the fuzzy set $\tilde{A}$. This set is denoted by core ($\tilde{A}$).

Definition 1.13. [6] A fuzzy set $\tilde{A}$ is said to be normal if $\sup_x \mu_{\tilde{A}}(x) = 1$.

Definition 1.14. [6] The membership function of the complement of a normalized fuzzy set $\tilde{A}$, $\mu_{\complement\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$ $x \in X$.

Definition 1.15. [6] For a finite fuzzy set $\tilde{A}$, the cardinality | $\tilde{A}$ | is defined as $|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)$. $\|\tilde{A}\| = |\tilde{A}| / |X|$ is called the relative cardinality of $\tilde{A}$.

Definition 1.16. [6] A fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$, $x_1, x_2 \in X, \lambda \in [0,1]$.

Definition 1.17. [1] Let A be a fuzzy set on U and α be a number such that $0 < \alpha \le 1$. Then by αA we mean a fuzzy set on U, denoted by αA which is such that $(\alpha A)(x) = \alpha A(x)$ for every x in U. This procedure of associating another fuzzy set with the given fuzzy set A is termed as restricted scalar multiplication.

Theorem 1.18. [1] Any fuzzy set A on U can be decomposed as A = sup $\{\alpha A_\alpha \;/ 0 < \alpha \le 1\}$. We also write $A = \sum \alpha A_\alpha$ or $A = \cup\ \alpha A_\alpha$.

In our previous paper of Uma Maheswari [4] in 2012, we introduced the new concept of fuzzy sets on the root systems of Kac- Moody algebras. The fuzzy set on $X = \Pi \times \Pi$ is defined as follows:

$$\mu_{\tilde{A}}(\alpha_i, \alpha_j) = \begin{cases} 1/\max(|a_{ij}|, |a_{ji}|) & if \quad a_{ij} \ne 0 \\ 0 & if \quad a_{ij} = 0 \end{cases} \qquad (1)$$

$$for\ (\alpha_i, \alpha_j) \in X, \ \ i, j = 1,2,\ldots, l$$

Then $\tilde{A} = ((\alpha_i, \alpha_j), (\mu_{\tilde{A}}(\alpha_i, \alpha_j))$ forms a fuzzy set on $\Pi \times \Pi$. .

The following properties of fuzzy sets defined by (1) on X , for finite type of Kac-Moody algebras are also given by Uma Maheswari [4] in 2012.

i) Support of $\tilde{A}$ consists of all (α_i, α_j) such that $a_{ij} \ne 0$, for $i, j = 1,2,\ldots, l$.

ii) Core of the fuzzy set $\tilde{A}$ is non – empty if and only if the associated Dynkin diagram contains at least one sub diagram of the form ○—○ .

iii) The fuzzy set $\tilde{A}$ defined by (1) is normal iff a sub diagram of the form ○—○ occurs in the Dynkin diagram associated with the GCM A.

2 Some fuzzy properties on the root systems of Affine Kac-Moody algebra

In this section we shall study about some more properties of fuzzy sets on the Cartesian product of the root basis of affine Kac-Moody algebras. The domain $X = \Pi \times \Pi$ is restricted to all elements $(\alpha_i, \alpha_j) \in \Pi \times \Pi$ which belong to Supp $\tilde{A}$, where $\mu_{\tilde{A}}(\alpha_i, \alpha_j) > 0$. For simplicity of notation, we shall represent the domain $X = \Pi \times \Pi$ itself.

Lemma 1. *A fuzzy set* $\tilde{A}$ *corresponding to the affine Kac-Moody algebras* $F_4^{(1)}, E_6^{(1)}, E_7^{(1)}$ & $E_8^{(1)}$ *defined by (1) are convex.*

Proof. Consider the affine type of Kac-Moody algebras $F_4^{(1)}, E_6^{(1)}, E_7^{(1)}$ & $E_8^{(1)}$. Let Π be the root basis for the corresponding affine Kac-Moody algebras listed here.

TABLE 1
ALL POSSIBLE MEMBERSHIP GRADE ATTAINED BY ANY ELEMENT IN X

$\mu_{\tilde{A}}(\alpha_i,\alpha_j)$	$\mu_{\tilde{A}}(\alpha_k,\alpha_l)$	$\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l))$	$\min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j), \mu_{\tilde{A}}(\alpha_k,\alpha_l)\}$
1	1	1	1
1	0	λ	0
1	1/2	$(\lambda+1)/2$	1/2
0	1	$1-\lambda$	0
0	0	0	0
0	1/2	$1-\lambda/2$	0
1/2	1	$(2-\lambda)/2$	1/2
1/2	0	$\lambda/2$	0
1/2	1/2	1/2	1/2

The table shows all possible membership grades for the elements of X and for every element in $X=\Pi\times\Pi$, we see that the following inequality is satisfied:
$\mu_{\tilde{A}}(\lambda x_1+(1-\lambda)x_2)\geq \min\{\mu_{\tilde{A}}(x_1),\mu_{\tilde{A}}(x_2)\},\ x_1,x_2\in X,\lambda\in[0,1].$
Hence the fuzzy sets $\tilde{A}$ corresponding to the affine type of Kac-Moody algebras $F_4^{(1)}, E_6^{(1)}, E_7^{(1)}\ \&\ E_8^{(1)}$ are convex.

Lemma 2.2. A fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $G_2^{(1)}$ defined by (1) is convex.

Proof. Consider the affine Kac-Moody algebra $G_2^{(1)}$. Let Π be the root basis for the corresponding affine Kac-Moody algebras listed here. The table showing all possible membership grades for the elements of X and the conditions for checking convexity is listed below:

TABLE 2
ALL POSSIBLE MEMBERSHIP GRADE ATTAINED BY ANY ELEMENT IN X

$\mu_{\tilde{A}}(\alpha_i,\alpha_j)$	$\mu_{\tilde{A}}(\alpha_k,\alpha_l)$	$\mu_{\tilde{A}}(\lambda(\alpha_i,\alpha_j)+(1-\lambda)(\alpha_k,\alpha_l))$	$\min\{\mu_{\tilde{A}}(\alpha_i,\alpha_j), \mu_{\tilde{A}}(\alpha_k,\alpha_l)\}$
1	1	1	1
1	0	λ	0
1	1/2	$(\lambda+1)/2$	1/2
1	1/3	$(2\lambda+1)/3$	1/3
0	1	$1-\lambda$	0
0	0	0	0
0	1/2	$(1-\lambda)/2$	0
0	1/3	$(1-\lambda)/3$	0
1/2	1	$(2-\lambda)/2$	1/2
1/2	0	$\lambda/2$	0
1/2	1/2	1/2	1/2
1/2	1/3	$(\lambda+2)/6$	1/3
1/3	1	$(3-2\lambda)/3$	1/3
1/3	0	$\lambda/3$	0
1/3	1/2	$(3-\lambda)/6$	1/3
1/3	1/3	1/3	1/3

For every element in $\Pi\times\Pi$, we see that the following inequality is satisfied:
$\mu_{\tilde{A}}(\lambda x_1+(1-\lambda)x_2)\geq \min\{\mu_{\tilde{A}}(x_1),\mu_{\tilde{A}}(x_2)\},\ x_1,x_2\in X,\lambda\in[0,1].$
Hence a fuzzy set $\tilde{A}$ corresponding to the affine type of Kac-Moody algebra $G_2^{(1)}$ is convex.

Computation of α – level sets :

We shall now determine the α –level sets and strong α – level sets for some specific cases of GCM, for the affine type of Kac-Moody algebras

$G_2^{(1)},\ F_4^{(1)},\ E_6^{(1)},\ E_7^{(1)}\ \&\ E_8^{(1)}.$

Theorem 2.3. *For the affine Kac-Moody algebra $G_2^{(1)}$ associated with the indecomposable GCM A, let $\tilde{A}$ be the fuzzy set defined on $\Pi\times\Pi$ given by the equation (1). Then the α - level sets and strong α - level sets for α = 1 , 1/2 , 1/3 ,...., 1/k,... are given below:*

$(i) A_1=\{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1)\}$

$(ii) A_{1/2}=A_1\cup\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3)\}$

$(iii) A_{1/3}=A_{1/2}\cup\{(\alpha_2,\alpha_3),(\alpha_3,\alpha_2)\}$

$(iv) A_{1/2}'=A_1$

$(v) A_{1/3}'=A_{1/2}$

$(vi) A_{1/4}'=A_{1/3}$

$(vii)\ \ |A_1|=2, |A_{1/2}|=5, |A_{1/3}|=7, |A_{1/4}'|=7$

For $k=4,5,\ldots,\ |A_{1/k}|=|A_{1/k}'|=7.$

Proof . Consider the family $G_2^{(1)}$

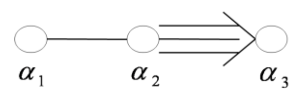

Fig. 1. Dynkin diagram for $G_2^{(1)}$

$(i) A_1=\{(\alpha_i,\alpha_j)\in X/\ \mu_{\tilde{A}}(\alpha_i,\alpha_j)\geq 1\}$

$=\{(\alpha_i,\alpha_j)\in X/\ \ (1/\max(|a_{ij}|,|a_{ji}|)\geq 1)\}=\{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1)\}$

$(ii) A_{1/2}=\{(\alpha_i,\alpha_j)\in X/\ \ (1/\max(|a_{ij}|,|a_{ji}|)\geq 1/2)\}$

$=A_1\cup\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3)\}$

$(iii) A_{1/3}=\{(\alpha_i,\alpha_j)\in X/\ \ (1/\max(|a_{ij}|,|a_{ji}|\geq 1/3)\}=A_{1/2}$

From the above relations we have,

$A_1\subset A_{1/2}=A_{1/3}=\ldots=A_{1/k}=\ldots$

$(iv) A_{1/2}'=\{(\alpha_i,\alpha_j)\in X/\ \ \mu_{\tilde{A}}(\alpha_i,\alpha_j)>1/2\}$

$=\{(\alpha_i,\alpha_j)\in X/\ \ (1/\max(|a_{ij}|,|a_{ji}|)>1/2)\}=A_1=\Phi$

$(v) A_{1/3}'=\{(\alpha_i,\alpha_j)\in X/\ \ 1/\max(|a_{ij}|,|a_{ji}|)>1/3\}=A_{1/2}$

$(vi) A_{1/4}'=\{(\alpha_i,\alpha_j)\in X/\ \ (1/\max(|a_{ij}|,|a_{ji}|)>1/4)\}=A_{1/3}$

From the above relations, we see that,

$A_1'\subset A_{1/2}'\subset A_{1/3}'\subset A_{1/4}'=\ldots=A_{1/k}'=\ldots$

(vii) By the above relations,

$$|A_1| = 2, |A_{1/2}| = 5, |A_{1/3}| = 7, |A_{1/4}'| = 7$$
$$A_{1/3} = \ldots = A_{1/k} = \ldots = A_{1/4}' = \ldots = A_{1/k}' = \ldots$$
$$For\ n = 4,5,\ldots,\ |A_{1/k}| = |A_{1/k}'| = 7.$$

Lemma 2.4. *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the finite type of Kac-Moody algebra $G_2^{(1)}$ by equation (1) then $\tilde{A}$ has the following properties:*

(a) The cardinality $|\tilde{A}| = 4.166$;

(b) Relative cardinality $\|\tilde{A}\| = 0.462$;

(c) The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the classical algebra $G_2^{(1)}$, for every $(\alpha_i, \alpha_j) \in X$ is listed below:

For $i = 2$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1}, \alpha_i) = 0$ & $\mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_{i-1}) = 0$;

For $i = 1,2,3$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_i) = 1/2$

For $i = 3$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_{i-1}) = 2/3$ *and*

$\mu_{\mathbb{C}\tilde{A}}(\alpha_3, \alpha_1) = \mu_{\mathbb{C}\tilde{A}}(\alpha_1, \alpha_3) = 1.$

Proof: (a) The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $G_2^{(1)}$ contains 2 elements in X having membership grade 1, 3 elements in X having membership grade 1/2 , 2 elements in X having membership grade 1/3 and all the other elements in X having membership grade 0. By the definition of cardinality,

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = 4.166$$

(b) $\|\tilde{A}\| = |\tilde{A}| / |X| = 0.462$

(c) The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $G_2^{(1)}$, is normal. For every $(\alpha_i, \alpha_j) \in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ is listed below,

For $i = 2$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1 = 0$;

For $i = 1,2,3$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_i) = 1 - 1/2 = 1/2$

For $i = 3$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_{i-1}) = 1 - 1/3 = 2/3$ *and*

$\mu_{\mathbb{C}\tilde{A}}(\alpha_3, \alpha_1) = \mu_{\mathbb{C}\tilde{A}}(\alpha_1, \alpha_3) = 1 - 0 = 1.$

Theorem 2.5. *For the affine Kac-Moody algebra $F_4^{(1)}$ associated with the indecomposable GCM A, let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ given by the equation (1). Then the α - level sets and strong α - level sets for α = 1 , 1/2 , 1/3 ,..., 1/k,... are given below:*

$(i) A_1 = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_4, \alpha_5), (\alpha_5, \alpha_4), (\alpha_3, \alpha_2), (\alpha_2, \alpha_1)\}$

$(ii) A_{1/2} = A_1 \cup \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_3, \alpha_3), (\alpha_4, \alpha_4), (\alpha_5, \alpha_5), (\alpha_3, \alpha_4), (\alpha_4, \alpha_3)\}$

$(iii) A_{1/2}' = A_1$

$(iv) A_{1/3}' = A_{1/2}$

$(v) |A_1| = 6, |A_{1/2}| = 13, |A_{1/3}'| = 13$

$$For\ k = 3,4,5,\ldots,\ |A_{1/k}| = |A_{1/k}'| = 13.$$

Proof . Consider the family $F_4^{(1)}$

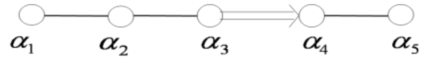

Fig. 2. Dynkin diagram for $F_4^{(1)}$

$(i) A_1 = \{(\alpha_i, \alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i, \alpha_j) \geq 1\}$

$= \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_4, \alpha_5), (\alpha_5, \alpha_4), (\alpha_3, \alpha_2), (\alpha_2, \alpha_1)\}$

$(ii) A_{1/2} = \{(\alpha_i, \alpha_j) \in X /\ (1/\max(|a_{ij}|, |a_{ji}|) \geq 1/2)\}$

$= A_1 \cup \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_3, \alpha_3), (\alpha_4, \alpha_4), (\alpha_5, \alpha_5), (\alpha_3, \alpha_4), (\alpha_4, \alpha_3)\}$

$A_{1/3} = \{(\alpha_i, \alpha_j) \in X /\ (1/\max(|a_{ij}|, |a_{ji}| \geq 1/3)\} = A_{1/2}$

From the above relations we have,

$$A_1 \subset A_{1/2} = A_{1/3} = \ldots = A_{1/k} = \ldots$$

$(iii) A_{1/2}' = \{(\alpha_i, \alpha_j) \in X /\ \mu_{\tilde{A}}(\alpha_i, \alpha_j) > 1/2\}$

$= \{(\alpha_i, \alpha_j) \in X /\ (1/\max(|a_{ij}|, |a_{ji}|) > 1/2)\} = A_1 = \Phi$

$(iv) A_{1/3}' = \{(\alpha_i, \alpha_j) \in X /\ 1/\max(|a_{ij}|, |a_{ji}|) > 1/3\} = A_{1/2}$

$A_{1/4}' = \{(\alpha_i, \alpha_j) \in X /\ (1/\max(|a_{ij}|, |a_{ji}|) > 1/4)\} = A_{1/3}$

From the above relations, we see that,

$$A_1' \subset A_{1/2}' \subset A_{1/3}' \subset A_{1/4}' = \ldots = A_{1/k}' = \ldots$$

(v) By the above relations,

$$A_{1/3} = \ldots = A_{1/k} = \ldots = A_{1/4}' = \ldots = A_{1/k}' = \ldots$$

$|A_1| = 6, |A_{1/2}| = 13, |A_{1/3}'| = 13$

$$For\ k = 3,4,5,\ldots,\ |A_{1/k}| = |A_{1/k}'| = 13.$$

Lemma 2.6. *Let $\tilde{A}$ be the fuzzy set defined on $\Pi \times \Pi$ for the affine Kac-Moody algebra $F_4^{(1)}$ by equation (1) then $\tilde{A}$ has the following properties:*

(a) The cardinality $|\tilde{A}| = 9.5$;

(b) Relative cardinality $\|\tilde{A}\| = 0.38$;

(c) The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $F_4^{(1)}$, for every $(\alpha_i, \alpha_j) \in X$ listed below:

For $i = 2,3,5$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1}, \alpha_i) = 0$ & $\mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_{i-1}) = 0$;

For $i = 1,2,3,4,5$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_i) = 1/2$

For $i = 3$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1}, \alpha_i) = \mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_{i-1}) = 2/3$ *and*

$\mu_{\mathbb{C}\tilde{A}}(\alpha_3, \alpha_1) = \mu_{\mathbb{C}\tilde{A}}(\alpha_1, \alpha_3) = 1.$

For $i \neq j,\ i \neq j+1$ & $j \neq i+1,$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_i, \alpha_j) = 1$ *and*

$\mu_{\mathbb{C}\tilde{A}}(\alpha_3, \alpha_4) = \mu_{\mathbb{C}\tilde{A}}(\alpha_4, \alpha_3) = 1/2.$

Proof. (a) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $F_4^{(1)}$ contains 6 elements in X having membership grade 1, 7 elements in X having membership grade 1/2 and all the other elements in X having membership grade 0.

By the definition of cardinality,

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = 9.5$$

(b) $\|\tilde{A}\| = |\tilde{A}| / |X| = 0.38$

(c) The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $F_4^{(1)}$, is normal. For for every $(\alpha_i,\alpha_j)\in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ are listed below,

For $i=2,3,5$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1},\alpha_i)=\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_{i-1})=1-1=0$;

For $i=1,2,3,4,5$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_i)=1-1/2=1/2$

For $i=3$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1},\alpha_i)=\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_{i-1})=1-1/3=2/3$

and $\mu_{\mathbb{C}\tilde{A}}(\alpha_3,\alpha_1)=\mu_{\mathbb{C}\tilde{A}}(\alpha_1,\alpha_3)=1-0=1.$

For $i\neq j,\ \ i\neq j+1\ \ \&\ \ j\neq i+1,$

$\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_j)=1-0=1$ and $\mu_{\mathbb{C}\tilde{A}}(\alpha_3,\alpha_4)=\mu_{\mathbb{C}\tilde{A}}(\alpha_4,\alpha_3)=1-1/2=1/2.$

Theorem 2.7. *For the affine Kac-Moody algebra $E_6^{(1)}$ associated with the indecomposable GCM A, let $\tilde{A}$ be the fuzzy set defined on $\Pi\times\Pi$ given by the equation (1). Then the α - level sets and strong α - level sets for α = 1 , 1/2 , 1/3 ,..., 1/k,... are given below:*

$(i) A_1=\{(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),(\alpha_3,\alpha_4),(\alpha_4,\alpha_5),(\alpha_3,\alpha_6),(\alpha_6,\alpha_7),$
$(\alpha_7,\alpha_6),(\alpha_6,\alpha_3)(\alpha_5,\alpha_4),(\alpha_4,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_1)\}$

$(ii) A_{1/2}=A_1\cup\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_5,\alpha_5),(\alpha_6,\alpha_6),(\alpha_7,\alpha_7)\}$

$(iii) A_{1/2}{}'=A_1$

$(iv) A_{1/3}{}'=A_{1/2}$

$(v)\ \ |A_1|=12, |A_{1/2}|=19, |A_{1/3}{}'|=19$

For $k=3,4,5,\ldots,\ |A_{1/k}|=|A_{1/k}{}'|=\ 19.$

Proof . Consider the family $E_6^{(1)}$

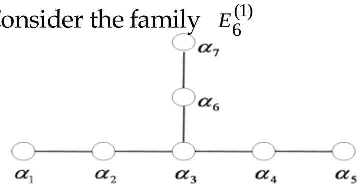

Fig. 3. Dynkin diagram for $E_6^{(1)}$

$(i) A_1=\{(\alpha_i,\alpha_j)\in X\ /\ \mu_{\tilde{A}}(\alpha_i,\alpha_j)\geq 1\}$
$=\{(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),(\alpha_3,\alpha_4),(\alpha_4,\alpha_5),(\alpha_3,\alpha_6),(\alpha_6,\alpha_7),$
$(\alpha_7,\alpha_6),(\alpha_6,\alpha_3)(\alpha_5,\alpha_4),(\alpha_4,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_1)\}$

$(ii) A_{1/2}=\{(\alpha_i,\alpha_j)\in X\ /\ \ (1/\max(|a_{ij}|,|a_{ji}|)\geq 1/2)\}$
$=A_1\cup\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_5,\alpha_5),(\alpha_6,\alpha_6),(\alpha_7,\alpha_7)\}$

$A_{1/3}=\{(\alpha_i,\alpha_j)\in X\ /\ \ (1/\max(|a_{ij}|,|a_{ji}|\geq 1/3)\}=A_{1/2}$

From the above relations we have,

$A_1\subset A_{1/2}=A_{1/3}=...=A_{1/l}=...$

$(iii) A_{1/2}{}'=\{(\alpha_i,\alpha_j)\in X\ /\ \ \mu_{\tilde{A}}(\alpha_i,\alpha_j)>1/2\}$
$=\{(\alpha_i,\alpha_j)\in X\ /\ \ (1/\max(|a_{ij}|,|a_{ji}|)>1/2)\}=A_1=\Phi$

$(iv) A_{1/3}{}'=\{(\alpha_i,\alpha_j)\in X\ /\ \ 1/\max(|a_{ij}|,|a_{ji}|)>1/3\}=A_{1/2}$

$A_{1/4}{}'=\{(\alpha_i,\alpha_j)\in X\ /\ \ (1/\max(|a_{ij}|,|a_{ji}|)>1/4)\}=A_{1/3}$

From the above relations, we see that,

$A_1{}'\subset A_{1/2}{}'\subset A_{1/3}{}'\subset A_{1/4}{}'=\ldots=A_{1/k}{}'=\ldots$

(v) By the above relations,

$A_{1/3}=\ldots=A_{1/k}=\ldots=A_{1/4}{}'=\ldots=A_{1/k}{}'=\ldots$

$|A_1|=12, |A_{1/2}|=19, |A_{1/3}{}'|=19$

For $k=3,4,5,\ldots,\ |A_{1/k}|=|A_{1/k}{}'|=\ 19.$

Lemma 2.8. *Let $\tilde{A}$ be the fuzzy set defined on $\Pi\times\Pi$ for the affine Kac-Moody algebra $E_6^{(1)}$ by equation (1) then $\tilde{A}$ has the following properties:*

(a) The cardinality $|\tilde{A}|$ = *15.5 ;*

(b) Relative cardinality $\|\tilde{A}\|$ = *0.316 ;*

(c) The membership function of the complement of a normalized fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_6^{(1)}$, for every $(\alpha_i,\alpha_j)\in X$ is listed below:

For $i=1,2,3,4,5,7$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1},\alpha_i)=0\ \ \&\ \ \mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_{i-1})=0$;

For $i=1,2,3,4,5,6,7$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_i)=1/2$

For $i\neq j,\ \ i\neq j+1\ \ \&\ \ j\neq i+1,$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_j)=1$ *and* $\mu_{\mathbb{C}\tilde{A}}(\alpha_3,\alpha_6)=\mu_{\mathbb{C}\tilde{A}}(\alpha_6,\alpha_3)=0.$

Proof. (a) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_6^{(1)}$ contains 12 elements in X having membership grade 1, 7 elements in X having membership grade 1/2 and all the other elements in X having membership grade 0.

By the definition of cardinality,

$$|\tilde{A}|=\sum_{x\in X}\mu_{\tilde{A}}(x)=15.5\,.$$

(b) $\|\tilde{A}\|=|\tilde{A}|/|X|=0.316$

(c) The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_6^{(1)}$, is normal. For every $(\alpha_i,\alpha_j)\in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ is listed below:

For $i=1,2,3,4,5,7$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_{i-1},\alpha_i)=\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_{i-1})=1-1=0$;

For $i=1,2,3,4,5,6,7$ $\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_i)=1-1/2=1/2$

For $i\neq j,\ \ i\neq j+1\ \ \&\ \ j\neq i+1,$ *implies*

$\mu_{\mathbb{C}\tilde{A}}(\alpha_i,\alpha_j)=1-0=1$ *and* $\mu_{\mathbb{C}\tilde{A}}(\alpha_3,\alpha_6)=\mu_{\mathbb{C}\tilde{A}}(\alpha_6,\alpha_3)=1-1=0.$

Theorem 2.9. *For the affine Kac-Moody algebra $E_7^{(1)}$ associated with the indecomposable GCM A, let $\tilde{A}$ be the fuzzy set defined on $\Pi\times\Pi$ given by the equation (1). Then the α - level sets and strong α - level sets for α = 1 , 1/2 , 1/3 ,..., 1/k,... are given below:*

$(i) A_1=\{(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),(\alpha_3,\alpha_4),(\alpha_4,\alpha_5),(\alpha_5,\alpha_6),(\alpha_6,\alpha_7),(\alpha_4,\alpha_8),$
$(\alpha_8,\alpha_4),(\alpha_7,\alpha_6),(\alpha_6,\alpha_5),(\alpha_5,\alpha_4),(\alpha_4,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_1)\}$

$(ii) A_{1/2}=A_1\cup\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_5,\alpha_5),(\alpha_6,\alpha_6),(\alpha_7,\alpha_7),(\alpha_8,\alpha_8)\}$

$(iii) A_{1/2}{}'=A_1$

$(iv) A_{1/3}{}'=A_{1/2}$

$(v) |A_1|=14, |A_{1/2}|=22, |A_{1/3}{}'|=22$

For $k=3,4,5,\ldots,\ |A_{1/k}|=|A_{1/k}{}'|=\ 22.$

Proof . Consider the family $E_7^{(1)}$

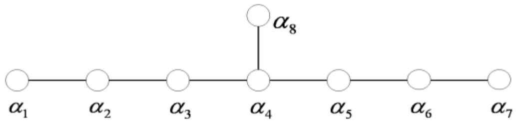

Fig. 4. Dynkin diagram for $E_7^{(1)}$

$(i) A_1 = \{(\alpha_i,\alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i,\alpha_j) \geq 1\}$

$= \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),(\alpha_3,\alpha_4),(\alpha_4,\alpha_5),(\alpha_5,\alpha_6),(\alpha_6,\alpha_7),(\alpha_4,\alpha_8),$
$(\alpha_8,\alpha_4),(\alpha_7,\alpha_6),(\alpha_6,\alpha_5),(\alpha_5,\alpha_4),(\alpha_4,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_1)\}$

$(ii) A_{1/2} = \{(\alpha_i,\alpha_j) \in X / \ (1/\max(|a_{ij}|,|a_{ji}|) \geq 1/2)\}$

$= A_1 \cup \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_5,\alpha_5),(\alpha_6,\alpha_6),(\alpha_7,\alpha_7),(\alpha_8,\alpha_8)\}$

$A_{1/3} = \{(\alpha_i,\alpha_j) \in X / \ (1/\max(|a_{ij}|,|a_{ji}| \geq 1/3)\} = A_{1/2}$

From the above relations we have,

$$A_1 \subset A_{1/2} = A_{1/3} = \ldots = A_{1/k} = \ldots$$

$(iii) A_{1/2}' = \{(\alpha_i,\alpha_j) \in X / \ \mu_{\tilde{A}}(\alpha_i,\alpha_j) > 1/2\}$

$= \{(\alpha_i,\alpha_j) \in X / \ (1/\max(|a_{ij}|,|a_{ji}|) > 1/2)\} = A_1 = \Phi$

$(iv) A_{1/3}' = \{(\alpha_i,\alpha_j) \in X / \ 1/\max(|a_{ij}|,|a_{ji}|) > 1/3\} = A_{1/2}$

$A_{1/4}' = \{(\alpha_i,\alpha_j) \in X / \ (1/\max(|a_{ij}|,|a_{ji}|) > 1/4)\} = A_{1/3}$

From the above relations, we see that,

$$A_1' \subset A_{1/2}' \subset A_{1/3}' \subset A_{1/4}' = \ldots = A_{1/k}' = \ldots$$

(v) By the above relations,

$$A_{1/3} = \ldots = A_{1/k} = \ldots = A_{1/4}' = \ldots = A_{1/k}' = \ldots$$

$|A_1| = 14, |A_{1/2}| = 22, |A_{1/3}'| = 22$

For $k = 3,4,5,\ldots, |A_{1/k}| = |A_{1/k}'| = 22.$

Lemma 2.10. *Let* $\tilde{A}$ *be the fuzzy set defined on* $\Pi \times \Pi$ *for the affine Kac-Moody algebra* $E_7^{(1)}$ *by equation (1) then* $\tilde{A}$ *has the following properties:*

(a) The cardinality $|\tilde{A}|$ = *18 ;*

(b) Relative cardinality $\|\tilde{A}\|$ = *0.281 ;*

(c) The membership function of the complement of a normalized fuzzy set $\tilde{A}$ *corresponding to the affine Kac-Moody algebra* $E_7^{(1)}$ *, for every* $(\alpha_i,\alpha_j) \in X$ *are is listed below:*

For $i = 2,\ldots,7$ $\mu_{\complement\tilde{A}}(\alpha_{i-1},\alpha_i) = 0 \ \& \ \mu_{\complement\tilde{A}}(\alpha_i,\alpha_{i-1}) = 0$;

For $i = 1,2,\ldots,8$ $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_i) = 1/2$,

For $i \neq j, \ i \neq j+1 \ \& \ j \neq i+1,$ *if* $i = 4$ *then* $j \neq 8,$ *if* $i = 8$ *then* $j \neq 4$, $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_j) = 1$ *and* $\mu_{\complement\tilde{A}}(\alpha_4,\alpha_8) = \mu_{\complement\tilde{A}}(\alpha_8,\alpha_4) = 0.$

Proof. (a) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_7^{(1)}$ contains 14 elements in X having membership grade 1, 8 elements in X having membership grade 1/2 and all the other elements in X having membership grade 0.

By the definition of cardinality,

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x) = 18$$

(b) $\|\tilde{A}\| = |\tilde{A}| / |X| = 0.281$

(c) The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_7^{(1)}$, is normal. For every $(\alpha_i,\alpha_j) \in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ is listed below:

For $i = 2,\ldots,7$ $\mu_{\complement\tilde{A}}(\alpha_{i-1},\alpha_i) = \mu_{\complement\tilde{A}}(\alpha_i,\alpha_{i-1}) = 1 - 1 = 0$;

For $i = 1,2,\ldots,8$ $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_i) = 1 - 1/2 = 1/2$

For $i \neq j, \ i \neq j+1 \ \& \ j \neq i+1,$ if $i = 4$ then $j \neq 8,$ if $i = 8$ then $j \neq 4$ $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_j) = 1 - 0 = 1$ and $\mu_{\complement\tilde{A}}(\alpha_4,\alpha_8) = \mu_{\complement\tilde{A}}(\alpha_8,\alpha_4) = 0.$

Theorem 2.11. *For the affine Kac-Moody algebra* $E_8^{(1)}$ *associated with the indecomposable GCM* A, *let* $\tilde{A}$ *be the fuzzy set defined on* $\Pi \times \Pi$ *given by the equation (1). Then the* α *-level sets and strong* α *- level sets for* α = 1 , 1/2 , 1/3 ,..., 1/k,... *are given below:*

$(i) A_1 = \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),(\alpha_3,\alpha_4),(\alpha_4,\alpha_5),(\alpha_5,\alpha_6),(\alpha_6,\alpha_7),(\alpha_7,\alpha_8),$
$(\alpha_6,\alpha_9),(\alpha_9,\alpha_6),(\alpha_8,\alpha_7),(\alpha_7,\alpha_6),(\alpha_6,\alpha_5),(\alpha_5,\alpha_4),(\alpha_4,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_1)\}$

$(ii) A_{1/2} = A_1 \cup \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_5,\alpha_5),(\alpha_6,\alpha_6),$
$(\alpha_7,\alpha_7),(\alpha_8,\alpha_8),(\alpha_9,\alpha_9)\}$

$(iii) A_{1/2}' = A_1$

$(iv) A_{1/3}' = A_{1/2}$

(v) $|A_1| = 16, |A_{1/2}| = 25, |A_{1/3}'| = 25$

For $k = 3,4,5,\ldots, |A_{1/k}| = |A_{1/k}'| = 25.$

Proof. *Consider the family* $E_8^{(1)}$

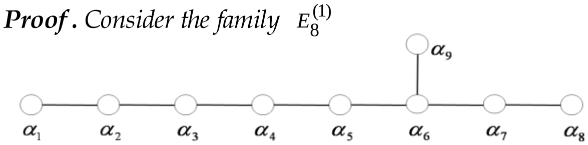

Fig. 5. Dynkin diagram for $E_8^{(1)}$

$(i) A_1 = \{(\alpha_i,\alpha_j) \in X / \mu_{\tilde{A}}(\alpha_i,\alpha_j) \geq 1\}$

$= \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),(\alpha_3,\alpha_4),(\alpha_4,\alpha_5),(\alpha_5,\alpha_6),(\alpha_6,\alpha_7),(\alpha_7,\alpha_8),$
$(\alpha_6,\alpha_9),(\alpha_9,\alpha_6),(\alpha_8,\alpha_7),(\alpha_7,\alpha_6),(\alpha_6,\alpha_5),(\alpha_5,\alpha_4),(\alpha_4,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_1)\}$

$(ii) A_{1/2} = \{(\alpha_i,\alpha_j) \in X / \ (1/\max(|a_{ij}|,|a_{ji}|) \geq 1/2)\}$

$= A_1 \cup \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_5,\alpha_5),(\alpha_6,\alpha_6),$
$(\alpha_7,\alpha_7),(\alpha_8,\alpha_8),(\alpha_9,\alpha_9)\}$

$A_{1/3} = \{(\alpha_i,\alpha_j) \in X / \ (1/\max(|a_{ij}|,|a_{ji}| \geq 1/3)\} = A_{1/2}$

From the above relations we have,

$$A_1 \subset A_{1/2} = A_{1/3} = \ldots = A_{1/k} = \ldots$$

$(iii) A_{1/2}' = \{(\alpha_i,\alpha_j) \in X / \ \mu_{\tilde{A}}(\alpha_i,\alpha_j) > 1/2\}$

$= \{(\alpha_i,\alpha_j) \in X / \ (1/\max(|a_{ij}|,|a_{ji}|) > 1/2)\} = A_1 = \Phi$

$(iv) A_{1/3}' = \{(\alpha_i,\alpha_j) \in X / \ 1/\max(|a_{ij}|,|a_{ji}|) > 1/3\} = A_{1/2}$

$A_{1/4}' = \{(\alpha_i,\alpha_j) \in X / \ (1/\max(|a_{ij}|,|a_{ji}|) > 1/4)\} = A_{1/3}$

From the above relations, we see that,

$$A_1' \subset A_{1/2}' \subset A_{1/3}' \subset A_{1/4}' = \ldots = A_{1/k}' = \ldots$$

(v) By the above relations,

$$A_{1/3} = \ldots = A_{1/l} = \ldots = A_{1/4}' = \ldots = A_{1/k}' = \ldots \ ;$$

$|A_1|=16, |A_{1/2}|=25, |A_{1/3}^{'}|=25$

For $k=3,4,5,\ldots, |A_{1/k}| = |A_{1/k}^{'}| = 25.$

Lemma 2.12. *Let* $\tilde{A}$ *be the fuzzy set defined on* $\Pi\times\Pi$ *for the affine Kac-Moody algebra* $E_8^{(1)}$ *by equation (1) then* $\tilde{A}$ *has the following properties*

(a) The cardinality $|\tilde{A}| = 20.5$ *;*

(b) Relative cardinality $\|\tilde{A}\| = 0.253$ *;*

(c) The membership function of the complement of a normalized fuzzy set $\tilde{A}$ *corresponding to the affine Kac-Moody algebra* $E_8^{(1)}$ *, for every* $(\alpha_i,\alpha_j)\in X$ *is listed below:*

For $i=2,\ldots,8$ $\mu_{\complement\tilde{A}}(\alpha_{i-1},\alpha_i)= \mu_{\complement\tilde{A}}(\alpha_i,\alpha_{i-1})=0$;

For $i=1,2,\ldots,9$ $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_i)=1/2$

For $i\neq j,\ i\neq j+1$ & $j\neq i+1,$ *if* $i=6$ *then* $j\neq 9,$ *if* $i=9$ *then* $j\neq 6$ $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_j)=1$ $\mu_{\complement\tilde{A}}(\alpha_6,\alpha_9)=\mu_{\complement\tilde{A}}(\alpha_9,\alpha_6)=0.$

Proof. (a) By the definition (1), the fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_8^{(1)}$ contains 16 elements in X having membership grade 1, 9 elements in X having membership grade 1/2 and all the other elements in X having membership grade 0.

By the definition of cardinality,

$$|\tilde{A}|=\sum_{x\in X}\mu_{\tilde{A}}(x)=20.5\,.$$

(b) $\|\tilde{A}\|=|\tilde{A}|/|X|=0.253$

(c) The fuzzy set $\tilde{A}$ corresponding to the affine Kac-Moody algebra $E_8^{(1)}$,is normal. For every $(\alpha_i,\alpha_j)\in X$, the membership function of the complement of a normalized fuzzy set $\tilde{A}$ is listed below:

For $i=2,\ldots,8$ $\mu_{\complement\tilde{A}}(\alpha_{i-1},\alpha_i)= \mu_{\complement\tilde{A}}(\alpha_i,\alpha_{i-1})=1-1=0$;

For $i=1,2,\ldots,9$ $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_i)=1-1/2=1/2;$

For $i\neq j,\ i\neq j+1$ & $j\neq i+1,$ if $i=6$ then $j\neq 9,$ if $i=9$ then $j\neq 6$ $\mu_{\complement\tilde{A}}(\alpha_i,\alpha_j)=1-0=1$ and $\mu_{\complement\tilde{A}}(\alpha_6,\alpha_9)=\mu_{\complement\tilde{A}}(\alpha_9,\alpha_6)=1-1=0.$

Theorem 2.13. *Let* $\tilde{A}$ *be the fuzzy set on* $\Pi\times\Pi$, *where* Π *denotes the root basis for the affine Kac-Moody algebras given by equation (1). Then the* α *- cut decomposition for the fuzzy set* $\tilde{A}$ *on the affine families* $G_2^{(1)}, F_4^{(1)}, E_6^{(1)}, E_7^{(1)}$ & $E_8^{(1)}$ *are*

(*i*) $G_2^{(1)}: 1A_1 \cup 1/2A_{1/2} \cup 1/3A_{1/3}.$

(*ii*) $F_4^{(1)}: A_1\subset A_{1/2};\ \tilde{A}=1A \cup 1/2A_{1/2}.$

(*iii*) $E_6^{(1)}: A=1A_1 \cup 1/2A_{1/2}.$

(*iv*) $E_7^{(1)}: A=1A_1 \cup 1/2A_{1/2}.$

(*v*) $E_8^{(1)}: A=1A_1 \cup 1/2A_{1/2}.$

Proof. (*i*) From Theorem 2.3, for the affine family $G_2^{(1)}$, $A_1\subset A_{1/2}\subset A_{1/3}$ and $A_{1/3}=X.$

By definition, the α – cut decomposition for the fuzzy set $\tilde{A}$ is $\cup\ \alpha A_\alpha$. Hence $\tilde{A}: 1A_1 \cup 1/2A_{1/2} \cup 1/3A_{1/3}.$

(*ii*) From Theorem 2.5, for the affine family $F_4^{(1)}$, $A_1\subset A_{1/2}$ and $A_{1/2}=X.$ Hence $\tilde{A}=1A_1 \cup 1/2A_{1/2}.$

(*iii*) From Theorem 2.7, for the affine family $E_6^{(1)}$, $A_1\subset A_{1/2}$ and $A_{1/2}=X.$ Hence $\tilde{A}=1A_1 \cup 1/2A_{1/2}.$

(*iv*) From Theorem 2.9, for the affine family $E_7^{(1)}$, $A_1\subset A_{1/2}$ and $A_{1/2}=X.$ Hence $\tilde{A}=1A_1 \cup 1/2A_{1/2}.$

(*v*) From Theorem 2.11, for the affine family $E_8^{(1)}$, $A_1\subset A_{1/2}$ and $A_{1/2}=X.$ Hence $\tilde{A}=1A_1 \cup 1/2A_{1/2}.$

3 Conclusion

We can further compute the level sets, determine the α-cut decomposition and study the structural properties for various families of affine , indefinite, hyperbolic, extended hyperbolic and non hyperbolic type of Kac-Moody algebras.

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